

(Reverse) engineering vacuum alignment

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ABSTRACT: In the presence of spontaneous symmetry breaking, the alignment of the vacuum with respect to the gauge group is often controlled by quadratically divergent operators in the low energy non-linear sigma model. In principle the magnitudes and signs of these operators can be changed by making different assumptions about the ultraviolet physics, but in practice all known ways of regulating these theories preserve the naïve vacuum alignment. We show that by “integrating in” different sets of heavy spin-one fields, it is possible to UV extend certain non-linear sigma models into two distinct UV insensitive theories. These UV extensions have identical low energy degrees of freedom but different radiative potentials, making it possible to engineer two different vacuum alignments for the original non-linear sigma model. Our construction employs “non-square” theory spaces which generically violate the common lore that the preferred vacuum alignment preserves the maximal gauge symmetry. By UV extending the $SO(9)/(SO(4) \times SO(5))$ little Higgs model, we find a radiative potential that deviates from the naïve expectation but does not stabilize the correct vacuum for proper electroweak symmetry breaking.

KEYWORDS: Chiral Lagrangians, Beyond Standard Model.

Contents

1. Motivation	1
2. The possibility of UV extension	3
3. Novel UV extensions and toggling	5
4. A simple example with UV insensitive toggling	9
5. UV extending the littlest Higgs with custodial symmetry	12
6. Future directions	15
A. Two-site non-square mooses	16
B. Parameters in the Simple Example	18
C. Vacuum alignment in doubly non-square mooses	19

1. Motivation

Because a non-linear sigma model (NL Σ M) describes degrees of freedom whose properties are defined entirely by spontaneous symmetry breaking, NL Σ Ms are ideal for understanding the low energy physics of QCD and technicolor, where strong dynamics obstructs calculability and only the symmetries are known *a priori*. Given the variety of possible symmetry breaking patterns, NL Σ Ms have also proven useful for building models with naturally light bosonic states. From the earliest theories of composite Higgs bosons [1, 2] to contemporary developments in little Higgs [3–10], T -parity [11–13], holographic Higgs [14, 15], and twin Higgs theories [16–18], NL Σ Ms have enriched our understanding of electroweak physics beyond the standard model.

However, the problem of vacuum alignment can hinder the construction of realistic models with the Higgs as a pseudo-Goldstone boson. Consider a theory with a global symmetry G spontaneously broken to a subgroup $H \subset G$. If G is an exact symmetry, then the orientation of H in G is arbitrary, and the Goldstone bosons in G/H parameterize a space of equivalent vacua. If a subgroup $F \subset G$ is weakly gauged, then gauge interactions specify a second orientation in G which lifts the vacuum degeneracy and chooses a preferred alignment of H relative to F . This vacuum alignment is crucial to phenomenology because it sets the particle content and charges of the low energy theory [19]. In both the $SU(6)/Sp(6)$ [6] and $SO(9)/(SO(4) \times SO(5))$ [9] little Higgs theories, a naïve calculation indicates that the stable vacuum does not allow for electroweak symmetry breaking.

Thus, it would be useful to have a method for engineering any desired vacuum alignment for a G/H NLSM with $F \subset G$ gauged. Of course, small tree level G -violating interactions can force such an alignment by hand, but we are looking for a method to fix the vacuum alignment from calculable one-loop corrections alone. In particular, when a NLSM suffers from one-loop quadratic divergences, one cannot trust the naïve stability of a given vacuum alignment because it is controlled by UV sensitive operators with incalculable coefficients. Several remedies have been proposed to render such operators calculable, all of which more or less involve the addition of new spin-one resonances. For example, if the theory is embedded in a higher dimensional spacetime, then what was formerly a UV sensitive operator becomes a nonlocal Wilson loop in the extra dimension. Since nonlocal operators cannot be quadratically sensitive, KK loops end up softening divergences [20]. Alternatively, one can invoke hidden local symmetry [21, 22], which introduces additional gauge bosons that cut off quadratically divergent loop integrals. As long as one assumes locality in theory space [23], then this cancelation will occur [24, 25].

But in both cases, the calculable vacuum alignment is the same as the vacuum alignment obtained from the naïve sign of quadratically divergent operators [22]. In QCD, the π^+ is heavier than the π^0 , just as one would “predict” from the quadratically divergent photon loop, and regulating QCD with a ρ meson in the vector limit [23] confirms that prediction [26]. There are examples like Casimir and thermal effects where the naïve signs of quadratically divergent operators are just plain incorrect. But in the context of vacuum alignment, we know of no examples (until now) where different vacuum alignments can be *chosen* by making different assumptions about ultraviolet physics.

In this paper, we present a novel method for UV extending low energy NLSMs when the global symmetry breaking pattern is

$$U(N) \rightarrow U(M) \times U(N - M). \quad (1.1)$$

There are two different ways of UV extending such NLSMs, and the two UV extensions are related by a binary “toggling” operation which keeps the light degrees of freedom fixed while reversing signs in the radiative potential. As we will see, the radiative potentials in these UV extensions are quadratically sensitive, but it is straightforward to further UV extend the NLSM into models that still exhibit toggling but are UV insensitive at leading order. Because the original NLSM and its UV extensions have different gauge structures, the resulting radiative potentials are qualitatively different. In contradiction to the common lore [19], the stable vacuum alignment in non-square UV extensions does not always preserve the maximal degree of gauge symmetry.

Toggling is only possible in “non-square” moose models. Until now, research on mooses has centered on “square” moose models, *i.e.* theory spaces in which every link field is a square matrix whose left and right global symmetries act on the same number of dimensions [27]. One reason for the popularity of square mooses is that they can be related to extra-dimensional theories via deconstruction [28], and square link fields often have natural UV completions as techni-fermion condensates. In contrast, because non-square mooses are crucial to the results of this paper, our construction has no obvious extra-dimensional

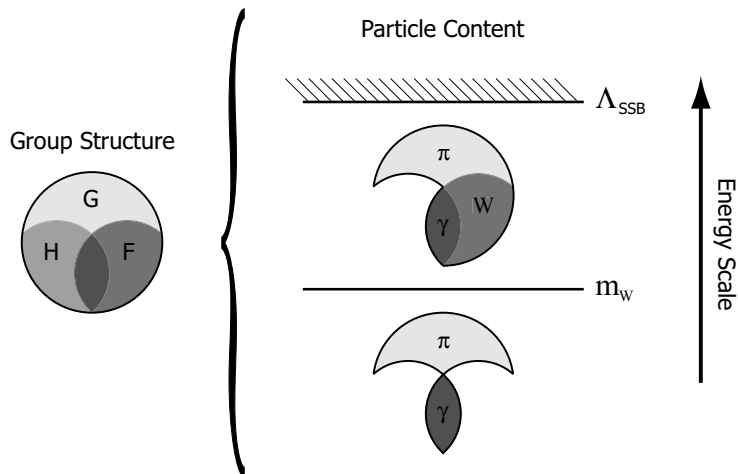


Figure 1: Ven diagrams representing the relative overlap of symmetry groups and particle multiplets. G , H , and F denote the global symmetry group, the unbroken global symmetry subgroup, and the gauge group, respectively. Using the language of technicolor, π , γ , and W denote the un-eaten pseudo-Goldstones, the massless gauge bosons, and the massive gauge bosons, respectively. At the scale m_W it is possible to produce W 's on-shell, and at Λ_{SSB} it is necessary to introduce a UV completion to unitarize the NLSM.

interpretation nor straightforward UV completion using strong dynamics. Also, our results depend crucially on theory space locality, which we assume throughout this paper.

In section 2, we discuss how two ultraviolet theories can have the same low energy degrees of freedom but qualitatively different radiative potentials. In section 3, we UV extend NLSMs based on eq. (1.1) into two-site mooses that exhibit toggling. Because these mooses are UV sensitive we cannot trust signs in the radiative potential, so in section 4 we present a simple example in which these theories are regulated into three-site mooses with finite one-loop potentials. As expected, toggling between these three-site UV extensions does reverse the signs of pseudo-Goldstone masses. In section 5, we study the $\text{SO}(9)/(\text{SO}(4) \times \text{SO}(5))$ little Higgs model in hopes of ameliorating its radiative instability, but discover that non-square mooses introduce a pseudo-Goldstone tadpole not present in the original potential. We conclude with some speculations about generalizing our construction to other NLSMs and UV completing non-square mooses. Details of our calculations and further examples are given in the appendices.

2. The possibility of UV extension

Before focusing on a specific class of NLSMs, we give a heuristic argument for how different UV assumptions can lead to different stable vacuum alignments. Consider a generic G/H NLSM with a subgroup $F \subset G$ weakly gauged. If we fix the alignment of F relative to H , then the theory consists of (pseudo-)Goldstone bosons in $G/(F \cup H)$ (“ π ”), massless gauge bosons in $F \cap H$ (“ γ ”), and massive gauge bosons in $F/(F \cap H)$ (“ W ”) which acquire longitudinal modes via the Higgs mechanism. We summarize the symmetries and degrees of freedom of this NLSM in figure 1. Because the W fields are the heaviest degrees of

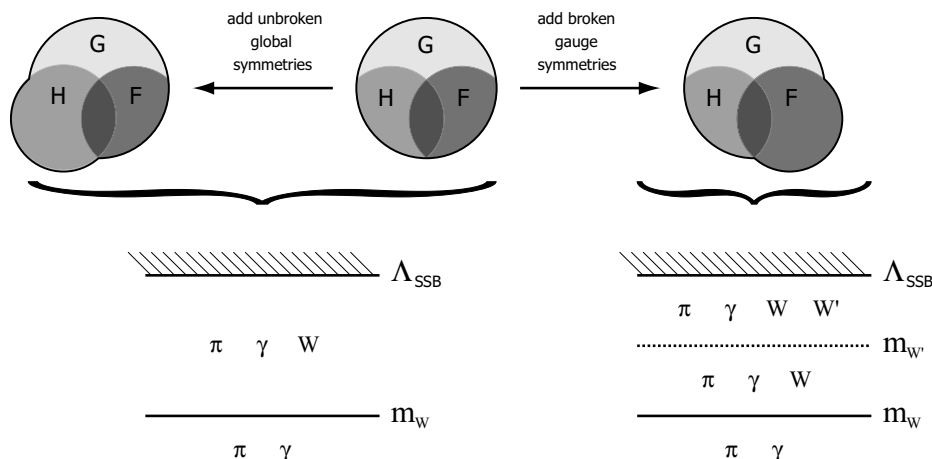


Figure 2: Starting from a low energy NLSM, one can generate new theories with identical infrared physics either by 1) adding (or subtracting) unbroken global symmetries, or 2) adding (or subtracting) broken gauge symmetries. While the former leaves the particle content unchanged all the way up to Λ_{SSB} , the latter requires the insertion of an intermediate scale, $m_{W'}$. While it is not necessary that $m_{W'} > m_W$, we are imagining that W' 's gauge coupling is large compared to W 's.

freedom, we can integrate them out at energies much lower than the symmetry breaking scale, leaving a low energy theory comprised of just π and γ . The alignment of F relative to H is stable if there are no tachyonic pseudo-Goldstone modes from radiative corrections.

Under what conditions will two different NLSMs share the same low energy degrees of freedom? If we are only interested in the dynamics of π and γ at tree level, then the specific choices G , H , and F are irrelevant as long as the low energy gauge group $F \cap H$ and uneaten Goldstone fields $G/(F \cup H)$ remain the same. This realization was used in [29] to show that the $SU(5)/SO(5)$ littlest Higgs model could be UV completed into ordinary QCD with five flavors, despite the fact that QCD has a chiral $SU(5)_L \times SU(5)_R$ global symmetry that is absent from the original model.

However, at one-loop level massive gauge bosons can affect the radiative potential for pseudo-Goldstone bosons in the theory, and depending on the specific choices of G , H , and F , there may or may not be tachyonic modes. If all we do is enlarge G by some new global symmetry that is left unbroken by the vacuum then the radiative potential will be the same because the gauge sector is unchanged. If, on the other hand, we introduce a new global symmetry that is fully gauged but maximally broken, then there is at least the possibility that the radiative potentials will be different. We call this a UV extension because the light degrees of freedom are fixed but the structure of the radiative potential can change in the presence of heavy gauge fields (see figure 2).

Our claim is that NLSMs based on $U(N) \rightarrow U(M) \times U(N - M)$ admit two possible UV extensions with different sets of heavy gauge bosons (see figure 3). As we describe in the following sections, the toggling operation that interchanges these two UV extensions can be used to manipulate signs in the radiative potential without changing the infrared particle content. In this way, different assumptions about heavy spin-one modes can lead to different stable vacuum alignments.

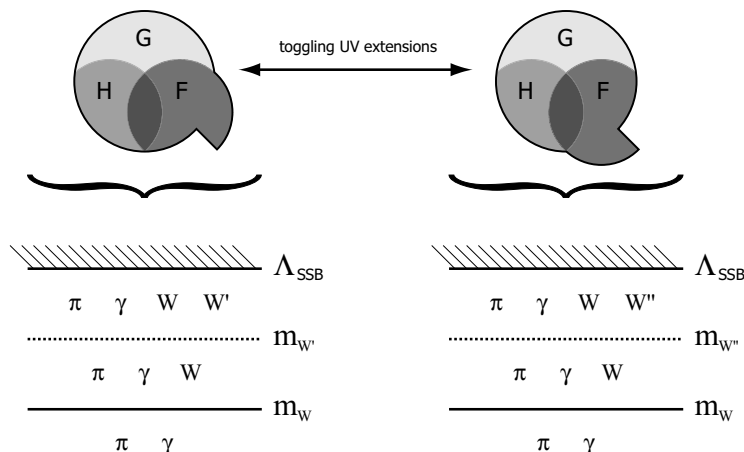


Figure 3: Toggling is a binary operation on theory space that interchanges two possible UV extensions of the original NL Σ M. Depending on whether we integrate in W 's or W'' 's above the scale m_W , we can generate either ultraviolet theory, and the choice of UV extension will affect the radiative stability of the vacuum.

3. Novel UV extensions and toggling

In this section we review little technicolor [29] or hidden local symmetry [21], which can be used to UV extend any NL Σ M. Then we show that for theories described by the symmetry breaking pattern in eq. (1.1), there exist two novel UV extensions using non-square mooses which are related by toggling. While these UV extensions suffer from quadratically sensitive operators, the naïve signs of these operators suggest that toggling should flip signs in the radiative potential. We will regulate this UV sensitivity in section 4 and show that toggling still occurs in one-loop finite theories.

Using little technicolor we can UV extend a G/H NL Σ M with $F \subset G$ gauged into a two-site moose given by

Global: G_L $\xrightarrow{\Sigma}$ G_R

Gauge: F H

	NL Σ M	Little Technicolor
\mathcal{G}	G	$G_L \times G_R$
\mathcal{H}	H	G_V
\mathcal{F}	F	$F \times H$

(3.1)

Because UV extending requires shuffling global and gauge symmetries, we use a slightly different notation from section 2. Here \mathcal{G} , \mathcal{H} , and \mathcal{F} denote the complete global, unbroken, and gauge symmetries of an entire theory, while G , H , and F denote specific groups.

The symmetry structure of the above theory is compactly represented by a moose diagram. Each site represents a symmetry group (global and gauge symmetries denoted above and below the site, respectively) while each link represents a matter field that transforms bifundamentally under those symmetries. For example, the link field Σ transforms as

$$\Sigma \rightarrow L \Sigma R^\dagger, \quad (L, R) \in G_L \times G_R. \tag{3.2}$$

As the result of spontaneous symmetry breaking, Σ acquires a vev equal to the identity matrix, leaving an unbroken symmetry G_V . In the limit that g_H , the gauge coupling for H ,

	NLSM	Little Technicolor
\mathcal{G}	$U(N)$	$U(N) \times U(N)$
\mathcal{H}	$U(M) \times U(N - M)$	$U(N)$
\mathcal{F}	\emptyset	$U(M) \times U(N - M)$
$\mathcal{G}/(\mathcal{F} \cup \mathcal{H})$	$U(N)/(U(M) \times U(N - M))$	$U(N)/(U(M) \times U(N - M))$
$\mathcal{F} \cap \mathcal{H}$	\emptyset	\emptyset
# of π 's	$2M(N - M)$	$2M(N - M)$
# of γ 's	0	0
# of W 's	0	$M^2 + (N - M)^2$

Table 1: Symmetries of the original NLSM and its little technicolor UV extension. Here we use the notation of eq. (3.1), where \mathcal{G} denotes the global symmetry group, \mathcal{H} denotes the unbroken global symmetry subgroup, and \mathcal{F} denotes the gauge group. Note that while little technicolor has more heavy gauge bosons compared to the NLSM, they have the same light degrees of freedom.

becomes large, the associated H gauge fields become ultra-massive and can be integrated out at low energies, yielding the original NLSM.¹ The heavy H gauge bosons cut off one-loop quadratic divergences in the radiative potential, and the qualitative structure of the regulated radiative potential is the same as that of the original NLSM [22]. Moreover, it is clear from the moose structure that little technicolor is the minimal deconstruction of an extra dimension with bulk gauge symmetry G bounded by two branes with reduced gauge symmetries F and H , as one would expect from the AdS/CFT correspondence [14].

While the square moose little technicolor construction is applicable to any NLSM, non-square moose UV extensions can be implemented for NLSMs of the form

$$\begin{aligned}
 \text{NLSM:} \quad G &= U(N), \\
 H &= U(M) \times U(N - M), \\
 F &= \emptyset.
 \end{aligned} \tag{3.3}$$

We have turned off gauge interactions F for simplicity, but as shown in appendix A, the following arguments still hold when they are included. Until now, the only known two-site UV extension for the NLSM in eq. (3.3) was

$$\text{Little Technicolor:} \quad \begin{array}{c} U(N) \\ \circ \end{array} \xrightarrow{\Sigma} \begin{array}{c} U(N) \\ \circ \end{array} \quad \langle \Sigma \rangle = \left(\mathbf{1}_{N \times N} \right), \tag{3.4}$$

$U(M) \times U(N - M)$

where $\mathbf{1}_{N \times N}$ denotes the $N \times N$ identity matrix. As shown in table 1, the original NLSM and the little technicolor UV extension have the same low energy degrees of freedom.

Non-square mooses permit two novel UV extensions of the original NLSM, denoted by Theory A and Theory B. Just like little technicolor, these UV extensions have the same

¹While integrating out the H gauge bosons will generate a series of higher dimensional operators (such as four fermion interactions and sigma field couplings), these terms will be suppressed by coefficients given by naïve dimensional analysis [30, 31] where factors of 4π are replaced by g_H . See [29] for details.

	Theory A	Theory B
\mathcal{G}	$U(N) \times U(M)$	$U(N) \times U(N - M)$
\mathcal{H}	$U(M) \times U(N - M)$	$U(M) \times U(N - M)$
\mathcal{F}	$U(M)$	$U(N - M)$
$\mathcal{G}/(\mathcal{F} \cup \mathcal{H})$	$U(N)/(U(M) \times U(N - M))$	$U(N)/(U(M) \times U(N - M))$
$\mathcal{F} \cap \mathcal{H}$	\emptyset	\emptyset
# of π 's	$2M(N - M)$	$2M(N - M)$
# of γ 's	0	0
# of W 's	M^2	$(N - M)^2$

Table 2: Symmetries of Theories A and B. Both of theories have the same infrared particle content as the original NL Σ M.

low energy Lagrangian as the original NL Σ M after integrating out the extra massive gauge bosons. The moose and vacuum structure of Theory A is

$$\text{Theory A: } \begin{array}{c} U(N) \\ \circ \\ \xrightarrow{\Sigma} \\ \circ \\ U(M) \end{array} \quad \langle \Sigma \rangle = \begin{pmatrix} \mathbf{1}_{M \times M} \\ \mathbf{0}_{(N-M) \times M} \end{pmatrix}, \quad (3.5)$$

where $\mathbf{0}_{(N-M) \times M}$ denotes the $(N - M) \times M$ zero matrix. We can generate Theory B by toggling Theory A. For any non-square moose, toggling has a two-fold effect: (a) it replaces a fully gauged symmetry at a site with a “conjugate” fully gauged symmetry, and (b) it replaces the vev of the neighboring link field with the “conjugate” vev. In particular, toggling sends Theory A to Theory B by replacing $U(M)$ with $U(N - M)$, giving

$$\text{Theory B: } \begin{array}{c} U(N) \\ \circ \\ \xrightarrow{\tilde{\Sigma}} \\ \circ \\ U(N - M) \end{array} \quad \langle \tilde{\Sigma} \rangle = \begin{pmatrix} \mathbf{0}_{M \times (N-M)} \\ \mathbf{1}_{(N-M) \times (N-M)} \end{pmatrix}. \quad (3.6)$$

As shown in table 2, Theories A and B also have same light degrees of freedom as the original NL Σ M, and this low energy equivalence is calculated explicitly in appendix A.

From the form of the vevs in eqs. (3.5) and (3.6), note that

$$\langle \tilde{\Sigma} \rangle \langle \tilde{\Sigma} \rangle^\dagger = 1 - \langle \Sigma \rangle \langle \Sigma \rangle^\dagger. \quad (3.7)$$

As shown in appendix A, in unitary gauge Σ and $\tilde{\Sigma}$ are written in terms of the *same* Goldstone matrix π , so

$$\Sigma = e^{i\pi/f} \langle \Sigma \rangle, \quad \tilde{\Sigma} = e^{i\pi/f} \langle \tilde{\Sigma} \rangle, \quad (3.8)$$

where f is the link field decay constant. Therefore, as far as the radiative potentials are concerned, toggling sends

$$\Sigma \Sigma^\dagger \rightarrow \tilde{\Sigma} \tilde{\Sigma}^\dagger = 1 - \Sigma \Sigma^\dagger, \quad (3.9)$$

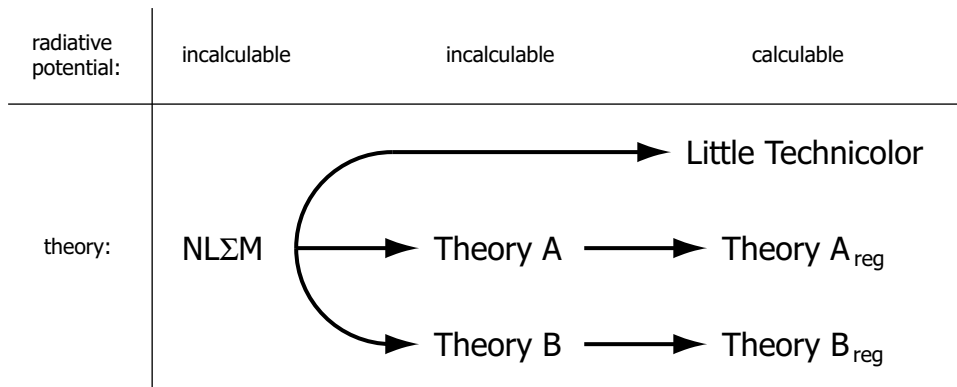


Figure 4: A summary of UV extensions of the original NLΣM, specifying moose structure and radiative stability of the naïve vacuum alignment. Because Theories A and B receive quadratically divergent radiative corrections, the signs of UV sensitive operators are incalculable. Nonetheless, these divergences can be rendered finite (and thus calculable) by UV extending into Theories A_{reg} and B_{reg} as discussed in section 4. In the little technicolor UV extension, the radiative potential is still calculable, albeit logarithmically divergent.

where the minus sign in the last term will be crucial to manipulating radiative potentials in the following sections.

In the presence of F gauge interactions, the light degrees of freedom in little technicolor and Theories A and B are exactly same as those in the original NLΣM with F gauged. While these theories have the same infrared particle content, they do not have the same radiative potentials since they have different gauge structures. In terms of the gauge boson mass matrix

$$(M^2)^{ab} = \frac{\partial^2 \mathcal{L}}{\partial A_\mu^a \partial A^{\mu b}}, \quad (3.10)$$

the one-loop Coleman-Weinberg radiative potential [32] is

$$V_{\text{CW}} = \frac{3\Lambda^2}{32\pi^2} \text{Tr}(M^2) + \frac{3}{64\pi^2} \text{Tr} \left(M^4 \log \frac{M^2}{\Lambda^2} \right). \quad (3.11)$$

Using this formula, both Theory A and Theory B have UV sensitive operators of the form

$$\Lambda^2 \text{Tr}(\Sigma \Sigma^\dagger C_F), \quad (3.12)$$

where $C_F = T_F^a T_F^a$ is the quadratic Casimir of F and $\Sigma \Sigma^\dagger$ is *not* the identity because Σ is not a square unitary matrix. Thus, strictly speaking the vacuum alignments in Theories A and B are unknown, as summarized in figure 4. Because eq. (3.12) could never arise in any square moose radiative potential, non-square theories have novel potentials compared to standard UV extensions.

Although the coefficient in front of eq. (3.12) should not be taken seriously, it is interesting to note that eq. (3.9) sends

$$\Lambda^2 \text{Tr}(\Sigma \Sigma^\dagger C_F) \rightarrow \Lambda^2 \text{Tr}(\tilde{\Sigma} \tilde{\Sigma}^\dagger C_F) = -\Lambda^2 \text{Tr}(\Sigma \Sigma^\dagger C_F) + \text{constant}. \quad (3.13)$$

Our naïve analysis indicates that modulo a constant, toggling flips the sign of this operator in the radiative potential. Physically, this difference occurs because the $U(M)$ and $U(N - M)$ gauge fields in Theories A and B form distinct subsets whose union comprises the full $U(M) \times U(N - M)$ of little technicolor. Since little technicolor has a constant one-loop quadratic divergence, the Σ dependent contributions from the $U(M)$ and $U(N - M)$ gauge multiplets must cancel, and thus Theories A and B have nontrivial quadratic divergences that differ only by a sign.² Roughly speaking, at the level of one-loop UV sensitive operators

$$V_{\text{Theory A}} + V_{\text{Theory B}} = V_{\text{Little Technicolor}}, \tag{3.14}$$

where the right hand side is precisely the constant in eq. (3.13).

One can still ask whether this result holds if we properly regulate these UV sensitive theories. In other words, if we UV extend Theories A and B into healthy, quadratic divergence-free theories, will toggling flip the signs of radiatively generated operators? The answer is yes, and we will consider a concrete example in the next section.

4. A simple example with UV insensitive toggling

Take the NL Σ M from the previous section with $N = 3$, $M = 2$, and a gauged $U(2)$ subgroup:

$$\begin{aligned} G &= U(3), \\ H &= U(2) \times U(1), \\ F &= U(2) \end{aligned} \tag{4.1}$$

This NL Σ M has roughly the same low energy degrees of freedom as the simple group little Higgs [7]. In section 3, we learned that this theory and its UV extensions, Theories A and B, exhibit quadratically divergent radiative corrections. In this section, we will regulate these UV sensitive operators by introducing even more heavy spin-one resonances. The resulting UV extensions, denoted by Theory A_{reg} and Theory B_{reg}, will have the same low energy degrees of freedom as the original NL Σ M, but qualitatively different radiative structures. Moreover, in accordance with the naïve prediction made in the previous section, these theories will have opposite radiative stability, as summarized in figure 5.

By inserting a fully gauged $U(3)$ site in the center of the moose in eq. (3.5), it is possible to regularize the quadratic divergences in Theory A:

$$\text{Theory A}_{\text{reg}}: \quad \begin{array}{ccc} \text{U(3)} & & \text{U(3)} & & \text{U(2)} \\ \circ & \xrightarrow{\Phi} & \circ & \xrightarrow{\Sigma} & \circ \\ \text{U(2)} & & \text{U(3)} & & \text{U(2)} \end{array} \tag{4.2}$$

$$\langle \Phi \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \langle \Sigma \rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.3}$$

²It is important to note that this property only holds for quadratically sensitive operators, and is a consequence of the fact that the Casimir of a product group $H = H_1 \times H_2$ is the sum of the Casimirs of H_1 and H_2 . For the calculable contributions to the radiative potential, eq. (3.14) no longer holds.

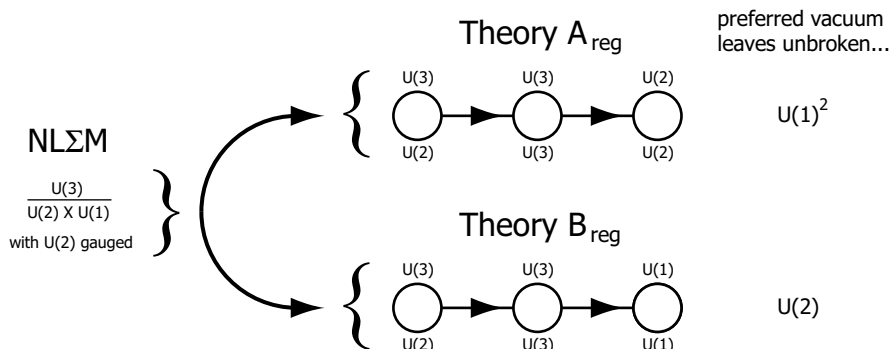


Figure 5: A summary of the regularized UV extensions of the NLSM in eq. (4.1), specifying moose structure and the gauge group preserved by the stable vacuum alignment. Interestingly, the stable vacuum for Theory A_{reg} is not that which preserves the maximal amount of gauge symmetry.

where Φ is an ordinary square link field. Likewise, Theory B in eq. (3.6) can be UV extended into

$$\text{Theory B}_{\text{reg}}: \quad \begin{array}{c} \text{U(3)} \\ \bigcirc \\ \text{U(2)} \end{array} \xrightarrow{\Phi} \begin{array}{c} \text{U(3)} \\ \bigcirc \\ \text{U(3)} \end{array} \xrightarrow{\tilde{\Sigma}} \begin{array}{c} \text{U(1)} \\ \bigcirc \\ \text{U(1)} \end{array} \quad (4.4)$$

$$\langle \Phi \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \langle \tilde{\Sigma} \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (4.5)$$

Because we can go to a gauge where all the Goldstones in Φ are eaten by the U(3) gauge group, it is obvious that Theories A_{reg} and B_{reg} have the same light degrees of freedom as Theories A and B and are therefore valid UV extensions of the original NLSM in eq. (4.1).

While the number of massive gauge bosons are different in these five theories, the light degrees of freedom are the same after integrating out heavy modes: they all contain a scalar doublet h charged under an unbroken U(2) gauge symmetry. In unitary gauge, the link fields in Theories A_{reg} and B_{reg} are

$$\Phi = e^{i\pi/f_{\text{eff}}} \langle \Phi \rangle, \quad \Sigma = e^{i\pi/f_{\text{eff}}} \langle \Sigma \rangle, \quad \tilde{\Sigma} = e^{i\pi/f_{\text{eff}}} \langle \tilde{\Sigma} \rangle, \quad (4.6)$$

where the Goldstone matrix is

$$\pi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & h \\ 0 & 0 & h \\ h^\dagger & 0 & 0 \end{pmatrix}, \quad (4.7)$$

and the effective pion decay constant is defined to properly normalize the Goldstone kinetic terms:

$$f_{\text{eff}} = \sqrt{f_\Phi^2 + \frac{f_\Sigma^2}{2}}. \quad (4.8)$$

It is now straightforward to calculate the radiative potentials for Theories A_{reg} and B_{reg} to see the effect of toggling. Since $\text{Tr}(M^2)$ and $\text{Tr}(M^4)$ are constants, both theories are completely free from quadratic and logarithmic divergences at one-loop.

The cancelation of divergences can be better understood in the language of spurions. In the absence of gauge couplings, Theory A_{reg} has three non-linearly realized $U(3)$ symmetries which protect the Goldstones from a mass: two acting on Φ from the left and right, and one acting on Σ from the left (the $U(2)$ acting on Σ from the right does not forbid mass terms). Denoting the gauge couplings on the left, middle and right gauge sites by g_F , g_G , and g_H , we see that g_F breaks the $U(3)$ acting on Φ from the left and g_G breaks the other two $U(3)$'s to the diagonal. Consequently, masses for the Goldstones must be generated at order $g_F g_G^2$.³ Note that this differs from little technicolor, which has only two $U(N)$ chiral symmetries protecting the Goldstone masses, and thus has logarithmically divergent radiative corrections.

To compute the finite Coleman-Weinberg radiative potential it is convenient to sum over vacuum bubbles in a background of pseudo-Goldstone bosons

$$V_{\text{CW}} = \frac{3}{2} \text{Tr} \int_0^\infty \frac{p^3 dp}{8\pi^2} \log \left(1 + \frac{M^2}{p^2} \right), \quad (4.9)$$

where M^2 is the gauge boson mass matrix and we have analytically continued into Euclidean space. Using the matrix identity $\text{Tr} \log X = \log \text{Det} X$ and expanding to quadratic order in the Goldstones, we find that to leading order in g_F^2 , the mass of the Higgs mode in Theory A_{reg} is

$$m_h^2 = -\frac{3}{32\pi^2} \frac{f_\Phi^2 f_\Sigma^2 g_F^2 g_G^2}{f_{\text{eff}}^2} \log(m_1^{\alpha_1} m_2^{\alpha_2} m_3^{\alpha_3}) + \mathcal{O}(g_F^4). \quad (4.10)$$

As detailed in appendix B, $m_1 > m_2 > m_3$ are physical gauge boson masses and α_i are coefficients that satisfy $\sum_i \alpha_i = 0$. For Theory B_{reg} we find

$$\tilde{m}_h^2 = \frac{3}{32\pi^2} \frac{f_\Phi^4 g_F^2 g_G^2}{f_{\text{eff}}^2} \log \left(\frac{\tilde{m}_1}{\tilde{m}_2} \right) + \mathcal{O}(g_F^4), \quad (4.11)$$

where $\tilde{m}_1 > \tilde{m}_2$. Note that \tilde{m}_h^2 is manifestly positive. Moreover, if we demand that the gauge couplings and pion decay constants are consistent with Theory A_{reg} being a UV extension of Theory A (i.e. such that new massive gauge bosons are heavier than pre-existing ones), then m_h^2 is negative. For example, given $f_1 = f_2 = f$ and $g_G = g_H = g$,

$$m_h^2 = \frac{f^2 g_F^2 g^2}{16\pi^2} \left(\log \left(\frac{3}{2} \right) + \frac{\sqrt{5}}{20} \log \left(\frac{47 - 21\sqrt{5}}{2} \right) \right) + \mathcal{O}(g_F^4) < 0, \quad (4.12)$$

$$\tilde{m}_h^2 = \frac{f^2 g_F^2 g^2}{16\pi^2} \log \left(\frac{3}{2} \right) + \mathcal{O}(g_F^4) > 0. \quad (4.13)$$

³In eqs. (4.10) and (4.11) below, the extra factor of g_G^2 comes from the fact that the α_i are proportional to g_G^2 . The fact that g_G controls the breaking of two different symmetries is reminiscent of collective breaking in the simple group little Higgs [7].

Thus $m_h^2 < 0 < \tilde{m}_h^2$, which is consistent with the naïve signs given by eq. (3.12) for Theories A and B. Our prediction from section 3 holds for Theories A_{reg} and B_{reg} and toggling does indeed flip signs in the radiative potential.

Alternatively, imagine starting with a low energy theory comprised of an h doublet and a $U(2)$ gauge symmetry. We can UV extend this theory into Theories A_{reg} and B_{reg} by integrating in heavy spin-one modes. Since these theories have different radiative potentials, we have the freedom to UV extend into whichever theory has the desired radiative stability. If we want a tachyonic doublet for “electroweak” symmetry breaking, then we would choose Theory A_{reg} , which radiatively generates a negative mass squared for h . If we instead want a stable doublet, we would choose Theory B_{reg} . In this way, it is possible to (reverse) engineer vacuum alignment by making different assumptions about the ultraviolet physics.

We have established that the vacuum alignment we chose for Theory A_{reg} is unstable, but what is the stable vacuum alignment? As it turns out, it is

$$\langle \Phi \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \langle \Sigma \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.14)$$

This stable vacuum alignment yields an unbroken $U(1)^2$ gauge symmetry, which has fewer generators than the unbroken $U(2)$ that results from the unstable vacuum alignment defined in eq. (4.3). Thus, despite the common lore, the vacuum does not necessarily align to preserve the maximal unbroken gauge symmetry. As we show in appendix C, non-square moose models violate this expectation quite generically.

5. UV extending the littlest Higgs with custodial symmetry

We have seen that for certain UV sensitive NLSMs, one can generate two different UV extensions with finite one-loop potentials. Moreover, toggling between these theories can flip signs in the radiative potential without changing the low energy degrees of freedom. Can this construction be used to stabilize vacuum alignments in phenomenologically interesting theories? Since the arguments made in section 3 and section 4 hold equally for orthogonal groups as well as as unitary groups, we can apply our results to the $SO(9)/(SO(4) \times SO(5))$ little Higgs model, described in [9].

In this theory, the vacuum spontaneously breaks a global $SO(9)$ to $SO(4) \times SO(5)$ with $SU(2)^3 \times U(1) \subset SO(9)$ gauged, leaving massless $SU(2)_L \times U(1)_Y$ gauge bosons and fourteen pseudo-Goldstone bosons at low energies. As originally formulated, this theory contains quadratically divergent operators whose naïve signs predict a saddle point in the radiative potential. If this theory is regulated using little technicolor, the calculable radiative potential reproduces the naïve expectation. For this reason it is interesting to ask whether a UV extension into non-square theory space might remedy this radiative instability, especially since the qualitative structure of the UV extended radiative potential has no relation to the original naïve potential. However, as we will see, non-square UV extensions of the $SO(9)/(SO(4) \times SO(5))$ little Higgs model not only fail to remove the saddle point, but exacerbate the situation by introducing a pseudo-Goldstone tadpole. The

tadpole points towards a stable vacuum that preserves all of F and is thus inconsistent with electroweak symmetry breaking.

The presence of a tadpole can be understood more broadly as a generic property of non-square UV extensions. These extensions generically suffer from tadpole instabilities unless the link field vevs and the quadratic Casimir of F satisfy a particular relationship. Consider two three-site UV extensions of a NL Σ M with the global symmetry breaking pattern of eq. (1.1). The first is defined analogously to eq. (4.2) by

$$\text{Theory A}_{\text{reg}}: \quad \begin{array}{c} \text{U(N)} \\ \circ \\ \text{F} \end{array} \xrightarrow{\Phi} \begin{array}{c} \text{U(N)} \\ \circ \\ \text{U(N)} \end{array} \xrightarrow{\Sigma} \begin{array}{c} \text{U(M)} \\ \circ \\ \text{U(M)} \end{array} \quad (5.1)$$

$$\langle \Phi \rangle = \left(\mathbf{1}_{N \times N} \right), \quad \langle \Sigma \rangle = \begin{pmatrix} \mathbf{1}_{M \times M} \\ \mathbf{0}_{(N-M) \times M} \end{pmatrix}, \quad (5.2)$$

and the second analogously to eq. (4.4) by

$$\text{Theory B}_{\text{reg}}: \quad \begin{array}{c} \text{U(N)} \\ \circ \\ \text{F} \end{array} \xrightarrow{\Phi} \begin{array}{c} \text{U(N)} \\ \circ \\ \text{U(N)} \end{array} \xrightarrow{\tilde{\Sigma}} \begin{array}{c} \text{U(N-M)} \\ \circ \\ \text{U(N-M)} \end{array} \quad (5.3)$$

$$\langle \Phi \rangle = \left(\mathbf{1}_{N \times N} \right), \quad \langle \tilde{\Sigma} \rangle = \begin{pmatrix} \mathbf{0}_{M \times (N-M)} \\ \mathbf{1}_{(N-M) \times (N-M)} \end{pmatrix}. \quad (5.4)$$

To ascertain whether a theory has a tadpole instability, one must compute the radiative potential. However, it is more elucidating to first consider the space of gauge invariant operators that can be generated at leading order in the gauge couplings. Since $\text{Tr}(M^2)$ and $\text{Tr}(M^4)$ are constant in both Theory A_{reg} and Theory B_{reg}, neither theory receives nontrivial radiative corrections at $\mathcal{O}(g^2)$ or $\mathcal{O}(g^4)$. At order $\mathcal{O}(g^6)$, there is only one allowed gauge invariant operator, namely

$$\text{Tr}(\Phi \Sigma \Sigma^\dagger \Phi^\dagger C_F) = \frac{2}{f_{\text{eff}}} \text{Tr}(\pi T) + \mathcal{O}(\pi^2) + \text{constant}, \quad (5.5)$$

where we are using the link field parametrization from eq. (4.6) and

$$T = i[\langle \Sigma \rangle \langle \Sigma \rangle^\dagger, \langle \Phi \rangle^\dagger C_F \langle \Phi \rangle]. \quad (5.6)$$

Since $\pi \in G/(F \cup H)$, this leading operator is tadpole-free if and only if the projection of T onto $G/(F \cup H)$ is zero, i.e. if

$$T|_{G/(F \cup H)} = 0. \quad (5.7)$$

The only effect of toggling is to flip the sign of T , so toggling has no effect on whether this criterion is satisfied. While it is not obvious whether theories that satisfy this constraint

are tadpole-free to all orders in the radiative potential, this determination can be made via an explicit evaluation of the Coleman-Weinberg potential.

For example, consider the $\text{SO}(9) \times (\text{SO}(4) \times \text{SO}(5))$ little Higgs theory, which can be UV extended into Theory A_{reg} or Theory B_{reg} if we replace unitary groups with orthogonal groups and take $N = 9$ and $M = 4$:

$$\begin{aligned} G &= \text{SO}(9), \\ H &= \text{SO}(4) \times \text{SO}(5), \\ F &= \text{SU}(2)^3 \times \text{U}(1). \end{aligned} \tag{5.8}$$

To evaluate eq. (4.9) and eq. (5.7), we use the Goldstone and gauge group embedding described in [9], rotated to a basis where the link field vev is diagonal. The fourteen pseudo-Goldstone bosons in this theory comprise a Higgs doublet \vec{h} , a singlet ψ^0 , and three triplets ψ^{ab} , where $a, b = 1, 2, 3$. We represent \vec{h} as a vector of $\text{SO}(4) \simeq \text{SU}(2)_L \times \text{SU}(2)_R$ where $\text{SU}(2)_R$ is a custodial symmetry, and group ψ^0 and ψ^{ab} into a four by four matrix

$$\Psi = \psi^0 \mathbf{1}_{4 \times 4} + 8\psi^{ab} T_L^a T_R^b, \tag{5.9}$$

where T_L^a and T_R^b are generators of $\text{SU}(2)_L$ and $\text{SU}(2)_R$. Going to unitary gauge, Theories A_{reg} and B_{reg} have link fields defined by eqs. (4.6) and (4.8), where

$$\pi = -\frac{i}{4} \begin{pmatrix} \mathbf{0}_{4 \times 4} & \Psi & 2\vec{h} \\ -\Psi & & \\ -2\vec{h}^T & \mathbf{0}_{5 \times 5} & \end{pmatrix}. \tag{5.10}$$

Also,

$$\langle \Sigma \rangle \langle \Sigma \rangle^\dagger = \begin{pmatrix} \mathbf{1}_{4 \times 4} & \mathbf{0}_{4 \times 5} \\ \mathbf{0}_{5 \times 4} & \mathbf{0}_{5 \times 5} \end{pmatrix}, \quad C_F = \frac{1}{8} \begin{pmatrix} \mathbf{5}_{4 \times 4} & -\mathbf{1}_{4 \times 4} & \mathbf{0}_{8 \times 1} \\ -\mathbf{1}_{4 \times 4} & \mathbf{5}_{4 \times 4} & \\ \mathbf{0}_{1 \times 8} & & 0 \end{pmatrix}. \tag{5.11}$$

Plugging these expressions into eq. (5.7), T has a component in the direction of ψ^0 , so the singlet has a tadpole. We also verified this result by evaluating the Coleman-Weinberg potential numerically.

A tadpole indicates that we are expanding around the wrong vacuum. Thus it is natural to ask, does the tadpole point towards a radiatively stable vacuum that is phenomenologically viable?⁴ Denoting the singlet generator by $\Delta \in G/(F \cup H)$, then rotating the vevs in this direction yields

$$\langle \Phi \rangle \rightarrow e^{i\Delta} \langle \Phi \rangle, \quad \langle \Sigma \rangle \rightarrow e^{i\Delta} \langle \Sigma \rangle, \quad \langle \tilde{\Sigma} \rangle \rightarrow e^{i\Delta} \langle \tilde{\Sigma} \rangle. \tag{5.12}$$

Computing eq. (5.5) for Theories A_{reg} and B_{reg} to quadratic order in the singlet,

$$\text{Tr}(\Phi \Sigma \Sigma^\dagger \Phi^\dagger C_F) = -\frac{\sin(\Delta)}{4} \left(\frac{\psi^0}{f_{\text{eff}}} - \cot(\Delta) \right)^2 + \text{constant}, \tag{5.13}$$

⁴Since the singlet transforms trivially under electroweak gauge transformations, we know that *any* vacuum rotated into the direction of the singlet preserves (at least) the $\text{SU}(2)_L \times \text{U}(1)_Y$ of the standard model. The only question is whether such a vacuum preserves *more* than this gauge symmetry.

for Theory A_{reg} and likewise for Theory B_{reg} except with an overall minus sign. Thus, the leading operator is tadpole-free if and only if

$$\Delta = \pi \left(n + \frac{1}{2} \right), \quad n \in \mathbb{Z} \tag{5.14}$$

From eq. (5.13) we see that ψ^0 has a potential consisting of alternating local minima and maxima, each of which leaves the entirety of F unbroken. Toggling between Theories A_{reg} and B_{reg} does little more than interchange minima and maxima, so neither theory has the appropriate low energy gauge group for successful electroweak symmetry breaking. Again, we verified these results beyond the leading order by computing the radiative potential numerically. Thus, we conclude that despite the control afforded by toggling, non-square UV extensions of the $SO(9) \times (SO(4) \times SO(5))$ little Higgs theory lack radiatively stable vacua consistent with electroweak physics.

6. Future directions

Understanding vacuum alignment is crucial for constructing realistic theories based on spontaneous symmetry breaking. While there is a large variety of symmetry breaking patterns available for model building, only certain vacuum alignments are radiatively stable in the presence of gauge interactions. We have shown that for a certain class of phenomenologically interesting NLSMs, different ultraviolet physics can yield different stable vacuum alignments in the infrared. Unfortunately, our technique fails to rectify vacuum instabilities in the $SO(9)/(SO(4) \times SO(5))$ little Higgs model [9].

Our result relies on using non-square mooses to UV extend NLSMs. Like the technique of hidden local symmetry, non-square UV extensions can have calculable radiative corrections, but because they lack an extra-dimensional interpretation, non-square mooses suggest new avenues for regulating more general NLSMs. As an interesting counterexample to the common lore, the stable vacuum does not necessarily preserve the maximal unbroken gauge symmetry in non-square mooses.

To what extent is it possible to engineer vacuum alignments in more general effective field theories, including those which lack a non-square moose representation? In section 2, we argued that any NLSM can be UV extended by adding (or subtracting) either unbroken global symmetries or fully gauged broken symmetries. Are there other realizations of this scenario? One might conjecture that non-square constructions could be generalized to any G/H NLSMs where H is a maximal subgroup of G .⁵ More phenomenologically relevant would be a method to UV extend the $SU(6)/Sp(6)$ little Higgs [6] to see whether the correct vacuum alignment for electroweak symmetry breaking could be ensured.

Another interesting question is how to UV complete non-square link fields. The theory space link fields become strongly coupled at $\Lambda \sim 4\pi f$ so new physics is needed to restore unitarity at that scale. Link fields can always be UV completed into linear sigma models, but can they arise from strong dynamics? For example, consider UV completing

⁵Indeed, the reason why we used $U(N)$ instead of $SU(N)$ groups is that a maximal subgroup of $SU(N)$ is $SU(N - M) \times SU(M) \times U(1)$. The extra $U(1)$ factor complicates toggling, though similar results hold.

a non-square link field with a fermion condensate $\langle \psi_i \psi_j^c \rangle$, where the ψ_i (ψ_j^c) transforms as a fundamental (anti-fundamental) under a confining group G_S . However, because the link field is non-square by assumption, the number of fundamental and anti-fundamental representations of G_S are different, generically introducing a gauge anomaly. One way of side-stepping this anomaly is to include spectator fermions also charged under G_S , a method considered for the SU(4)/SU(3) sigma models in the simple group little Higgs model [7]. Alternatively, for non-square mooses with SO(N) flavor symmetries, one might look for a G_S that exhibits confinement from two different real representations.

Finally, in this paper we have considered vacuum alignment in the presence of gauge fields alone. In any realistic composite Higgs theory, there will be additional contributions from fermion loops, and because of the large top Yukawa coupling, fermion loops can give the dominant contribution to the radiative potential. Because toggling changes the gauge structure of the theory it will also change the allowed fermion representations, so the action of toggling on the fermion sector is not well-defined. Then again, for little Higgs theories in particular, the challenge to making fully realistic theories is less the sign of fermion radiative corrections as the magnitude of those corrections, and generically, one needs some level of fine-tuning to get the correct electroweak scale [33]. Still, the fact that non-square mooses have such counter-intuitive properties inspires us to search for other novel mechanisms to adjust not only the sign but perhaps the magnitude of NL Σ M radiative potentials.

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A. Two-site non-square mooses

In this appendix, we show that the little technicolor construction and Theories A and B from section 3 have the same light degrees of freedom as the original NL Σ M.

The little technicolor theory is defined in eq. (3.4). The global symmetry breaking pattern is $U(N)_L \times U(N)_R \rightarrow U(N)_V$, which generates a $U(N)$'s worth of Goldstone bosons. Since the Goldstone modes parameterize the broken global symmetry directions, we define $\Sigma = L \langle \Sigma \rangle R^\dagger$, where $(L, R) \in U(N)_L \times U(N)_R$. The link field can be written as

$$\Sigma = U \langle \Sigma \rangle, \tag{A.1}$$

for some unitary matrix U , which can in turn be expressed as the product of an element of $H = U(M) \times U(N - M)$ and an element of $U(N)/H$

$$U = e^{i\pi/f} e^{ih/f}, \quad \pi = \begin{pmatrix} 0 & b \\ b^\dagger & 0 \end{pmatrix}, \quad h = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \tag{A.2}$$

where a and c are Hermitian $M \times M$ and $(N - M) \times (N - M)$ Goldstone matrices and b is a general complex $M \times (N - M)$ Goldstone matrix. Since the Σ vev breaks all of H , we

can go to a unitary gauge where the h Goldstones are eaten, leaving

$$U = e^{i\pi/f}. \tag{A.3}$$

A straightforward calculation shows that the Lagrangian for the little technicolor theory is

$$\mathcal{L} = -\frac{1}{2g_F^2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2g_H^2} \text{Tr}(H_{\mu\nu}H^{\mu\nu}) + f^2 \text{Tr} |G_\mu + H_\mu|^2, \tag{A.4}$$

$$G_\mu = iU^\dagger D_\mu U, \quad D_\mu = \partial_\mu + iF_\mu, \tag{A.5}$$

where F_μ and $F_{\mu\nu}$ denote the gauge field and field strength tensor for the gauge group F , and analogously for H . For later convenience we define

$$G_\mu = \begin{pmatrix} A_\mu & B_\mu \\ B_\mu^\dagger & C_\mu \end{pmatrix}. \tag{A.6}$$

If we take the gauge coupling $g_H \rightarrow \infty$, then the H_μ gauge bosons become ultra-massive and can be integrated out by setting them to their equations of motion. Since H_μ has no kinetic term in this limit, it acts as a Lagrange multiplier which effectively eliminates A_μ and C_μ , leaving the effective Lagrangian

$$\mathcal{L} = -\frac{1}{2g_F^2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + f^2 \text{Tr} \left| \begin{pmatrix} 0 & B_\mu \\ B_\mu^\dagger & 0 \end{pmatrix} \right|^2, \tag{A.7}$$

$$= -\frac{1}{2g_F^2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + f^2 \text{Tr}(p^\mu p_\mu), \tag{A.8}$$

where p_μ is the component of G_μ that falls in the $U(N)/H$ direction. Recall that eq. (A.8) is simply the CCWZ Lagrangian [34, 35] for a $U(N)/H$ NLSM with $F \subset U(N)$ gauged. We have arrived at the known result that little technicolor reproduces the original NLSM at low energies. For finite g_H , the Lagrangian is the same as eq. (A.8) with higher dimension operators suppressed by the mass of the heavy H gauge bosons [29].

Next, consider Theory A defined in eq. (3.5). The global symmetry breaking pattern is $U(N)_L \times U(M)_R \rightarrow U(M)_V \times U(N-M)_L$, yielding a $U(N)/U(N-M)$'s worth of Goldstones. Like before, we define $\Sigma = L\langle\Sigma\rangle R^\dagger$, although this time $(L, R) \in U(N)_L \times U(M)_R$. Since $\langle\Sigma\rangle$ is a N by M matrix where $N > M$, we can “pull” R^\dagger through the vev and write $\Sigma = L\bar{R}^\dagger\langle\Sigma\rangle$, where \bar{R}^\dagger is defined by

$$\bar{R} = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.9}$$

In general, if Σ has more rows than columns, it is always possible to write it as

$$\Sigma = U\langle\Sigma\rangle, \tag{A.10}$$

where $U = L\bar{R}^\dagger$. If Σ has fewer rows than columns, then this parametrization can instead be made for Σ^\dagger .

We perform the same coset decomposition of Σ as in eq. (A.2). In this parametrization c is eliminated immediately by the vev and a is eaten by the $H = U(M)$ gauge bosons, leaving precisely eq. (A.3). Since the symmetries corresponding to c are left unbroken by the vacuum, there are no propagating c Goldstone bosons. Thus, we have shown that Theory A has precisely the same Goldstone content as little technicolor and the original NLSM, albeit through a slightly different mechanism.

The Lagrangian for Theory A is

$$\mathcal{L} = -\frac{1}{2g_F^2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2g_H^2} \text{Tr}(H_{\mu\nu}H^{\mu\nu}) + f^2 \text{Tr}|G_\mu + \overline{H}_\mu|^2 P_\Sigma, \quad (\text{A.11})$$

where G_μ is defined as before, $P_\Sigma = \langle \Sigma \rangle \langle \Sigma \rangle^\dagger$, and \overline{H}_μ is an $N \times N$ matrix with H_μ in the upper $M \times M$ block and zeroes in the lower $(N - M) \times (N - M)$ block. By sending $g_H \rightarrow \infty$ and integrating out H_μ , we eliminate just the A_μ component of G_μ . Thus, the effective Lagrangian is

$$\mathcal{L} = -\frac{1}{2g_F^2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + f^2 \text{Tr} \left| \begin{pmatrix} 0 & B_\mu \\ B_\mu^\dagger & C_\mu \end{pmatrix} \right|^2 P_\Sigma, \quad (\text{A.12})$$

$$= -\frac{1}{2g_F^2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2} f^2 \text{Tr}(p^\mu p_\mu). \quad (\text{A.13})$$

Note that P_Σ eliminates the C_μ component from the Lagrangian. Having again generated the CCWZ Lagrangian, we see that little technicolor and Theory A have identical low energy physics up to a factor of $\sqrt{2}$ in the pion decay constant. The higher dimensional operators generated from integrating out H_μ with finite g_H are generically different from little technicolor.

A completely analogous computation can be done for Theory B, defined in eq. (3.6). We can express Σ in the form of eq. (A.2), although this time c is eliminated by the vev and a is eaten. After going to unitary gauge the link field can be written as

$$\tilde{\Sigma} = U \langle \tilde{\Sigma} \rangle \quad (\text{A.14})$$

where U is the same as in eq. (A.3). Thus, we conclude that the uneaten Goldstones in Theories A and B are exactly the same. Moreover, by integrating out massive gauge bosons we find that the effective Lagrangian for Theory B is again eq. (A.13).

B. Parameters in the Simple Example

Here, we define the parameters in the radiative potentials for Theories A_{reg} and B_{reg} from section 4. To leading order in g_F , the gauge boson masses in Theory A_{reg} are

$$m_1^2 = s + \sqrt{s^2 - t^2} + \mathcal{O}(g_F^2), \quad (\text{B.1})$$

$$m_2^2 = \frac{1}{2} f_\Phi^2 g_G^2 + \frac{1}{4} f_\Sigma^2 g_G^2, \quad (\text{B.2})$$

$$m_3^2 = s - \sqrt{s^2 - t^2} + \mathcal{O}(g_F^2), \quad (\text{B.3})$$

where

$$s = \frac{1}{4}f_\Phi^2 g_G^2 + \frac{1}{4}f_\Sigma^2 g_G^2 + \frac{1}{4}f_\Sigma^2 g_H^2, \quad t = \frac{1}{2}f_\Phi f_\Sigma g_G g_H. \quad (\text{B.4})$$

Note that $m_1^2 > m_2^2 > m_3^2$ for small enough g_F . The coefficients α_i are defined as

$$\alpha_1 = \frac{1}{4} \frac{m_1^2 f_\Phi^2 g_G^2 + 2t^2}{(m_1^2 - m_2^2)(m_1^2 - m_3^2)}, \quad (\text{B.5})$$

and similarly for α_2 and α_3 . For Theory B_{reg}, we define

$$\tilde{m}_1^2 = \frac{1}{2}f_\Phi^2 g_G^2 + \frac{1}{4}f_\Sigma^2 g_G^2, \quad (\text{B.6})$$

$$\tilde{m}_2^2 = \frac{1}{2}f_\Phi^2 g_G^2 + \mathcal{O}(g_F^2), \quad (\text{B.7})$$

and for small enough g_F , $\tilde{m}_1^2 > \tilde{m}_2^2$.

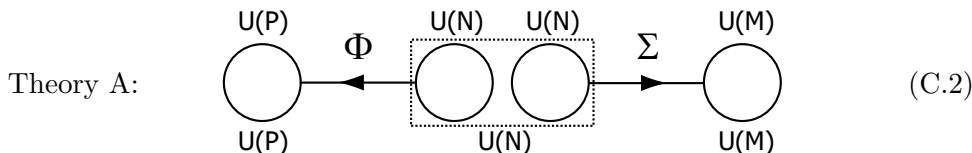
C. Vacuum alignment in doubly non-square mooses

For NLΣMs of the form

$$\begin{aligned} G &= \text{U}(N)_L \times \text{U}(N)_R, \\ H &= \text{U}(P)_L \times \text{U}(N - P)_L \times \text{U}(M)_R \times \text{U}(N - M)_R, \\ F &= \text{U}(N)_V, \end{aligned} \quad (\text{C.1})$$

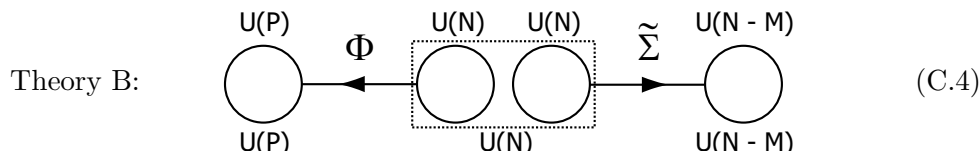
where $N > M, P$, it is possible to understand toggling and vacuum alignment completely analytically.⁶ Such a theory is a natural generalization of the simple group little Higgs [7], which apart from extra U(1) factors is defined by $N = 3$, and $M = P = 1$.

It is possible to UV extend the NLΣM in eq. (C.1) into a UV insensitive theory by using two non-square link fields:



$$\langle \Sigma \rangle = \begin{pmatrix} \mathbf{1}_{M \times M} \\ \mathbf{0}_{(N-M) \times M} \end{pmatrix}. \quad (\text{C.3})$$

The gauge couplings (g_P , g_N , and g_M) and the pion decay constants (f_Σ and f_Φ) can take arbitrary values. We can toggle Theory A to Theory B by sending $\text{U}(M) \rightarrow \text{U}(N - M)$



⁶If M or P is greater than N , then all Goldstone bosons are eaten and there is no meaning to vacuum alignment.

$$\langle \tilde{\Sigma} \rangle = \begin{pmatrix} \mathbf{0}_{M \times (N-M)} \\ \mathbf{1}_{(N-M) \times (N-M)} \end{pmatrix}, \quad (\text{C.5})$$

yielding a second UV extension. By toggling the $U(P)$ site it is possible to generate two additional UV extensions, Theories C and D. It is straightforward to see that all four theories have the same low energy degrees of freedom; they are comprised of a pair of two-site mooses with the diagonal $U(N)$ weakly gauged, so we can simply invoke the low energy equivalence between two-site mooses established in section 3.

The stable vacuum alignment for each of these theories can be determined from the radiative potential from eq. (3.11). By simply writing down the leading gauge invariant operators, we see immediately that the leading order contribution comes at $\mathcal{O}(g^4)$ and is

$$\text{Tr} |\Phi^\dagger \Sigma|^2. \quad (\text{C.6})$$

Evaluating eq. (3.11) explicitly for Theory A yields⁷

$$V_{\text{CW}} = -\frac{3N}{256\pi^2} f_\Phi^2 f_\Sigma^2 g_N^4 \log \left(\frac{\Lambda^2}{m^2} \right) \text{Tr} |\Phi^\dagger \Sigma|^2, \quad (\text{C.7})$$

where m is the scale of masses for the heavy gauge fields.

What is the stable vacuum alignment for non-square mooses? For convenience, we construct projection operators from the link field vevs by

$$P_\Phi = \langle \Phi \rangle \langle \Phi \rangle^\dagger, \quad P_\Sigma = \langle \Sigma \rangle \langle \Sigma \rangle^\dagger. \quad (\text{C.8})$$

Using the Goldstone parametrization of eq. (4.6), we can go to a unitary gauge where

$$\pi = \{P_\Phi, \pi\} = \{P_\Sigma, \pi\}. \quad (\text{C.9})$$

Expanding $V_{\text{CW}} \sim -f_\Phi^2 f_\Sigma^2 \text{Tr} |\Phi^\dagger \Sigma|^2$ to quadratic order in the Goldstones:

$$V_{\text{CW}} \sim -f_\Phi^2 f_\Sigma^2 \text{Tr} \left(P_\Phi e^{i\pi/f_{\text{eff}}} P_\Sigma e^{-i\pi/f_{\text{eff}}} \right), \quad (\text{C.10})$$

$$\sim (f_\Phi^2 f_\Sigma^2 / f_{\text{eff}}^2) \text{Tr} (\pi^2 (2P_\Phi P_\Sigma - P_\Sigma)) + \mathcal{O}(\pi^3), \quad (\text{C.11})$$

$$\sim (f_\Phi^2 f_\Sigma^2 / f_{\text{eff}}^2) \text{Tr} (\pi^2 (2P_\Phi P_\Sigma - P_\Phi)) + \mathcal{O}(\pi^3). \quad (\text{C.12})$$

Thus, the theory is tachyon-free if and only if

$$P_\Phi P_\Sigma = P_\Sigma \quad \text{or} \quad P_\Phi P_\Sigma = P_\Phi. \quad (\text{C.13})$$

Immediately we see that the more “aligned” the vevs are, the more likely that the projection operators constructed from them will satisfy the tachyon-free constraint.

We can use eq. (C.13) to determine the stable vacuum alignment in generic non-square mooses. For the theory in eq. (C.2) we can assume $N > M \geq P$ without loss of generality.

⁷Interestingly, the logarithmic divergence is proportional to the $U(N)$ gauge coupling only. In this way, non-square mooses are a generalization of the simple group method for achieving collective symmetry breaking [7].

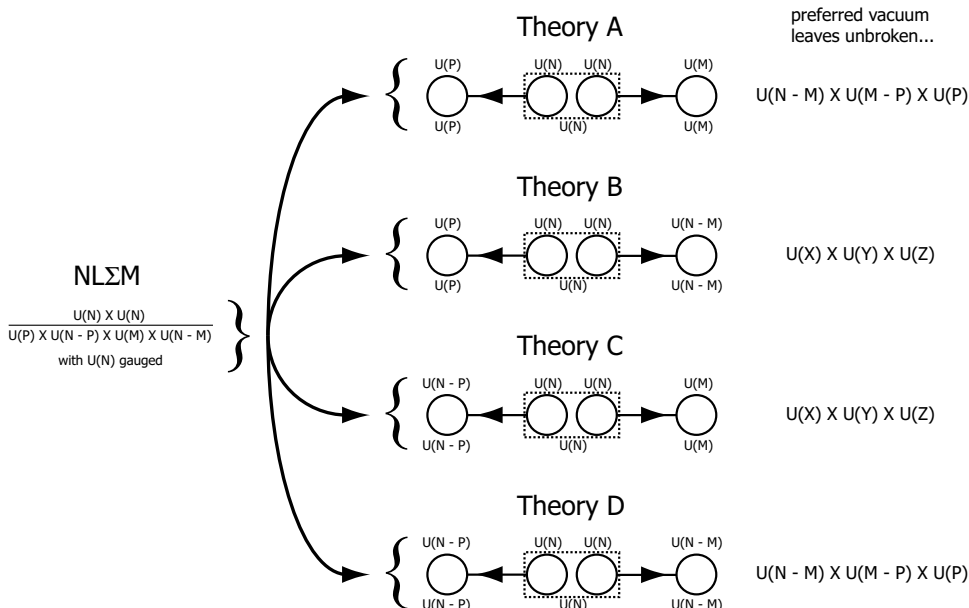


Figure 6: Given the NLΣM defined in eq. (C.1), one can generate four different UV extensions based on non-square mooses. The four theories yield two different stable vacuum alignments, and the stable vacuum is the one for which the link field vevs are maximally parallel. Without loss of generality, we assume $N > M \geq P$, and we define $X = \min(M, N - P)$, $Y = \min(P, N - M)$ and $Z = |N - M - P|$

If the theory is to be tachyon-free, then P_Φ must lie entirely within P_Σ . Consequently, the stable vacuum alignment leaves

$$\mathcal{F} \cap \mathcal{H} = U(N - M) \times U(M - P) \times U(P) \tag{C.14}$$

as the unbroken gauge symmetry.

Putting together all these results, there are four different ways to UV extend the NLΣM in eq. (C.1) using non-square mooses. All four are related to Theory A in eq. (C.2) by toggling either Φ or Σ or both. As summarized in figure 6, these four mooses yield two different stable vacuum alignments: one is the naïve vacuum alignment “predicted” by [19] and the other is a novel vacuum alignment from the perspective of the original NLΣM.

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